

The lower bound to the concurrence for four-qubit W state under noisy channels

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(Dated: January 7, 2015)

Abstract

We study the dynamics of four-qubit W state under various noisy environments by solving analytically the master equation in the Lindblad form in which the Lindblad operators correspond to the Pauli matrices and describe the decoherence of states. Also, we investigate the dynamics of the entanglement using the lower bound to the concurrence. It is found that while the entanglement decreases monotonically for Pauli-Z noise, it decays suddenly for other three noises. Moreover, by studying the time evolution of entanglement of various maximally entangled four-qubit states, we indicate that the four-qubit W state is more robust under same-axis Pauli channels. Furthermore, three-qubit W state preserves more entanglement with respect to the four-qubit W state, except for the Pauli-Z noise.

PACS numbers: 03.65.Yz, 05.40.Ca, 03.67.Mn

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I. INTRODUCTION

The theory of open systems has an important role in quantum information science [1]. The interaction between the system and the environment which leads to the relaxation and decoherence is the origin of the classical states. Therefore, studying open quantum systems can help us to protect the quantum states using proper methods [2–7]. The application of quantum dynamics of open systems ranges from quantum cosmology to quantum optics and to quantum information [8–15]. For instance, the detailed coherent control over components of quantum systems is one of the greatest challenges in quantum information processing. Among these efforts, the control of entanglement against the dissipative effects of the environment is an important issue.

The dynamics of open quantum systems is usually described by the quantum Markov process as a semigroup of completely positive dynamical maps and the corresponding quantum master equation in the Lindblad form [16, 17]. However, there are many quantum systems which display non-Markovian behavior in which correlations can give rise to memory effects due to the environment [18–20]. The Lindblad equation as the most general Markov evolution equation for a density operator is given by [16]

$$\frac{\partial \rho}{\partial t} = -\frac{i}{\hbar}[H_S, \rho] + \frac{1}{2}([L, \rho L^\dagger] + [L\rho, L^\dagger]), \quad (1)$$

where the Lindblad operator L represents the influence of the environment.

Since the entanglement is known as a resource for performance enhancements in quantum information processing, understanding and control of the entanglement in open systems have many applications in quantum computation and quantum information. As the first attempt to study the dynamics of entanglement in open systems, the evolution of entanglement has been studied using the state changes rather than computing the entanglement of an arbitrary state by an entanglement measure [21]. The first real entanglement dynamics in open systems investigated the dynamics of entanglement for a pair initially entangled harmonic oscillators under the action of local environments [22]. Also, the evolution of the average entanglement of formation of random bipartite states has been analyzed in Ref. [23]. In these works, the authors showed that the entanglement vanishes at finite times and coherence decayed asymptotically at long times.

The first study of dynamics of multipartite systems explored multiqubit systems under the influence of single-qubit depolarization and examined the robustness of three- and four-

qubit GHZ, W, and Dicke states and found that GHZ states are more robust than other generic states [24]. In Ref. [25], in addition to depolarizing noise, the effects of Pauli noises are studied on three-qubit GHZ and W states and discovered that three-qubit GHZ state preserves more entanglement than three-qubit W state. The dynamical evolution of N -qubit GHZ and W states for $2 \leq N \leq 7$ coupling to the independent dephasing and thermal baths at zero temperature and infinite temperature has been compared numerically using multipartite concurrence as a quantifier of the entanglement. It is observed that the decay rate increases with N for the GHZ state. For the W state this phenomenon happens only for infinite temperature environment. Also, it is size-independent for dephasing and zero-temperature thermal reservoirs [26]. In Ref. [27], the robustness of N -qubit GHZ state as resources for teleportation against Pauli channels and dephasing channel has been studied. It is found that three-qubit GHZ state is more robust than N -qubit ($N > 3$) GHZ states under most noisy channels. It is also shown that three-qubit W state is more robust than three-qubit GHZ state for small noisy parameter while the GHZ state becomes more robust when the noisy parameter is large [28]. Moreover, for W state-like superpositions against dephasing and amplitude damping channels, using three measures of entanglement, it is shown that the effects of decoherence on the fidelity is better described by the Meyer-Wallach global entanglement measure [29].

In this paper, we study the changes of entanglement for the initial four-qubit standard W state which will be mixed due to transmission through noisy channels. For this purpose, we prepare initially a pure W state then by solving analytically Eq. (1) with Pauli matrices as the Lindblad operators we obtain dynamical evolution of system under the influence of Pauli channels as well as isotropic (depolarizing) channel. Also, to compute the entanglement, we employ a lower bound for the multi-qubit concurrence proposed by Li *et al.* [30]. We note that the W state with the form

$$|W_N\rangle = \frac{1}{\sqrt{N}} (|00 \dots 01\rangle + |00 \dots 10\rangle + \dots + |01 \dots 00\rangle + |10 \dots 00\rangle), \quad (2)$$

is an example of multipartite entangled states in which N denotes the number of qubits in the state. Next, in order to compare the robustness¹ of W_4 state with maximally entangled four-partite states, we examine the behavior of entanglement for three four-qubit maximally

¹ The notion of “robustness” here and throughout the paper is the persistence of the lower bound of entanglement in noisy environments.

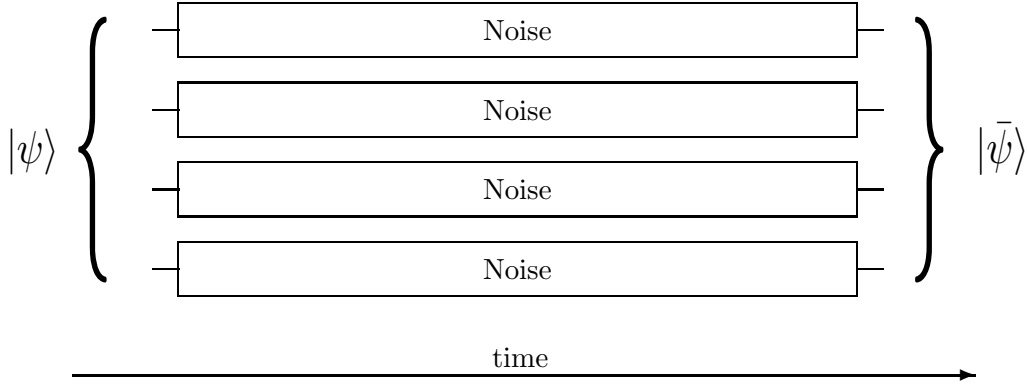


FIG. 1: A schematic quantum circuit for the transmission of a four-qubit state through a noisy environment.

entangled states that are given by [31]

$$|\phi_1\rangle = \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle), \quad (3)$$

$$|\phi_2\rangle = \frac{1}{2} (|1111\rangle + |1100\rangle + |0010\rangle + |0001\rangle), \quad (4)$$

$$|\phi_3\rangle = \frac{1}{\sqrt{6}} \left(\sqrt{2}|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle \right). \quad (5)$$

The rest of paper is as follows: In the next section, we analytically solve the Lindblad equation for the initial four-qubit W state and study the evolution of state under some Markovian noises. Also, we study the behavior of entanglement of states under the Pauli noises. In Sec. III, we investigate the time evolution of entanglement for maximally entangled four-qubit states. In the last section, we present our conclusions.

II. EVOLUTION OF ENTANGLEMENT OF W_4 STATE UNDER NOISY CHANNELS

In this section, we analytically solve the Lindblad equation (1) for the initial W_4 state in contact with some Markovian noises and study the dynamics of the entanglement of the states. It is remarkable to point that the Lindblad equation for a four-particle state consists of 136 coupled differential equations which in principle their solution is a difficult task. Here, using the evolution of the density matrix at small times, we propose a proper ansatz for the sought-after density matrix which leads to at most 11 coupled equations that is much simpler

to solve. Fig. (1) shows the quantum circuit that describes schematically the effect of noise on the state.

For the first case, consider Pauli-X (bit-flip) channel. When a four-qubit W state transmits through this noise, Eq. (1) leads to 16 diagonal and 120 off-diagonal coupled linear differential equations. Thus, to simplify the problem, we first consider the time evolution of the density matrix W_4 state for infinitesimal time interval δt as

$$\rho(\delta t) = \rho(0) + \left[\sum_{i=1}^4 \left(L_{i,x} \rho(0) L_{i,x}^\dagger \right) - \frac{1}{2} \left\{ L_{i,x}^\dagger L_{i,x}, \rho(0) \right\} \right] \delta t, \quad (6)$$

in which $L_{i,x} = \sqrt{\kappa_{i,x}} \sigma_x^i$ and

$$\rho(0) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (7)$$

Substituting $\rho(0)$ in Eq. (6) gives

$$\rho_{W_4}^x(\delta t) = \frac{1}{4} \begin{pmatrix} 4\kappa\delta t & 0 & 0 & 2\kappa\delta t & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 2\kappa\delta t & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 0 & \kappa\delta t & \kappa\delta t & 0 & 0 & \kappa\delta t & \kappa\delta t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & \kappa\delta t & 0 & 2\kappa\delta t & \kappa\delta t & 0 & 0 & \kappa\delta t & 0 & 0 & \kappa\delta t & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & \kappa\delta t & 0 & \kappa\delta t & 2\kappa\delta t & 0 & 0 & 0 & \kappa\delta t & 0 & \kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & \kappa\delta t & 0 & \kappa\delta t & 0 & 0 & 0 & 2\kappa\delta t & \kappa\delta t & 0 & \kappa\delta t & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & \kappa\delta t & 0 & 0 & \kappa\delta t & 0 & 0 & \kappa\delta t & 2\kappa\delta t & 0 & \kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2\kappa\delta t & 0 & 0 & 0 & 0 & \kappa\delta t & \kappa\delta t & 0 & 0 & \kappa\delta t & \kappa\delta t & 0 & 2\kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (8)$$

Now, consider the following ansatz for all times which is consistent with $\rho_{W_4}^x(\delta t)$:

$$\rho_{W_4}^x(t) = \begin{pmatrix} a & 0 & 0 & d & 0 & d & d & 0 & 0 & d & d & 0 & d & 0 & 0 & d \\ 0 & b & c & 0 & c & 0 & 0 & p & c & 0 & 0 & p & 0 & p & 0 & 0 \\ 0 & c & b & 0 & c & 0 & 0 & p & c & 0 & 0 & p & 0 & 0 & p & 0 \\ d & 0 & 0 & e & 0 & m & m & 0 & 0 & m & m & 0 & 0 & 0 & 0 & q \\ 0 & c & c & 0 & b & 0 & 0 & p & c & 0 & 0 & 0 & 0 & p & p & 0 \\ d & 0 & 0 & m & 0 & e & m & 0 & 0 & m & 0 & 0 & m & 0 & 0 & q \\ d & 0 & 0 & m & 0 & m & e & 0 & 0 & 0 & m & 0 & m & 0 & 0 & q \\ 0 & p & p & 0 & p & 0 & 0 & f & 0 & 0 & 0 & n & 0 & n & n & 0 \\ 0 & c & c & 0 & c & 0 & 0 & 0 & b & 0 & 0 & p & 0 & p & p & 0 \\ d & 0 & 0 & m & 0 & m & 0 & 0 & 0 & e & m & 0 & m & 0 & 0 & q \\ d & 0 & 0 & m & 0 & 0 & m & 0 & 0 & m & e & 0 & m & 0 & 0 & q \\ 0 & p & p & 0 & 0 & 0 & 0 & n & p & 0 & 0 & f & 0 & n & n & 0 \\ d & 0 & 0 & 0 & 0 & m & m & 0 & 0 & m & m & 0 & e & 0 & 0 & q \\ 0 & p & 0 & 0 & p & 0 & 0 & n & p & 0 & 0 & n & 0 & f & n & 0 \\ 0 & 0 & p & 0 & p & 0 & 0 & n & p & 0 & 0 & n & 0 & n & f & 0 \\ 0 & 0 & 0 & q & 0 & q & q & 0 & 0 & q & q & 0 & q & 0 & 0 & h \end{pmatrix}. \quad (9)$$

Inserting this solution in the Lindblad equation results in a set of 11 coupled differential equations

$$\begin{cases} \dot{a}(t) = 4k(b(t) - a(t)), \\ \dot{b}(t) = k(a(t) - 4b(t) + 3c(t)), \\ \dot{c}(t) = 2k(d(t) + m(t) - 2c(t)), \\ \dot{d}(t) = 2k(c(t) + p(t) - 2d(t)), \\ \dot{e}(t) = 2k(b(t) + f(t) - 2e(t)), \\ \dot{f}(t) = k(3e(t) + h(t) - 4f(t)), \\ \dot{h}(t) = 4k(f(t) - h(t)), \\ \dot{p}(t) = k(d(t) + 2m(t) - 4p(t) + q(t)), \\ \dot{m}(t) = k(c(t) - 4m(t) + n(t) + 2p(t)), \\ \dot{n}(t) = 2k(m(t) + q(t) - 2n(t)), \\ \dot{q}(t) = 2k(n(t) + p(t) - 2q(t)), \end{cases} \quad (10)$$

subject to the initial conditions $b(0) = c(0) = 1/4$ and $a(0) = d(0) = e(0) = f(0) = h(0) = p(0) = m(0) = n(0) = q(0) = 0$. The solutions read

$$\begin{cases} a(t) = 2d(t) = \frac{1}{16} (1 + 2e^{-2\kappa t} - 2e^{-6\kappa t} - e^{-8\kappa t}), \\ b(t) = \frac{1}{16} (1 + e^{-2\kappa t} + e^{-6\kappa t} + e^{-8\kappa t}), \\ c(t) = \frac{1}{32} (1 + 2e^{-2\kappa t} + 2e^{-4\kappa t} + 2e^{-6\kappa t} + e^{-8\kappa t}), \\ e(t) = 2m(t) = \frac{1}{16} (1 - e^{-8\kappa t}), \\ f(t) = \frac{1}{16} (1 - e^{-2\kappa t} - e^{-6\kappa t} + e^{-8\kappa t}), \\ h(t) = 2q(t) = \frac{1}{16} (1 - 2e^{-2\kappa t} + 2e^{-6\kappa t} - e^{-8\kappa t}), \\ n(t) = \frac{1}{32} (1 - 2e^{-2\kappa t} + 2e^{-4\kappa t} - 2e^{-6\kappa t} + e^{-8\kappa t}), \\ p(t) = \frac{1}{32} (1 - 2e^{-4\kappa t} + e^{-8\kappa t}). \end{cases} \quad (11)$$

Now, using $\rho_{W_4}^x(t)$ we examine the evolution of the entanglement for this state. As a quantifier of the entanglement, we use the lower bound to multipartite concurrence which is introduced by Li *et al.* [30]

$$C_N(\rho) \geq \tau_N(\rho) \equiv \sqrt{\frac{1}{N} \sum_{n=1}^N \sum_{k=1}^K (C_k^n)^2}, \quad (12)$$

that it is given in terms of N bipartite concurrences C^n correspond to the possible bipartite cuts of the multi-qubit system in which one of qubits is separated from the other qubits.

The bipartite concurrence C^n is defined by a sum of $K = 2^{N-2}(2^{N-1} - 1)$ terms which are expressed as $C_k^n = \max\{0, \lambda_k^1 - \lambda_k^2 - \lambda_k^3 - \lambda_k^4\}$, where λ_k^m , $m = 1, \dots, 4$, are the square roots of four nonvanishing eigenvalues in decreasing order of the matrix $\rho \tilde{\rho}_k^n$. Here, $\tilde{\rho}_k^n = S_k^n \rho^* S_k^n$ and S_k^n are defined by the tensor product of the generators of $\text{SO}(2)$ and $\text{SO}(2^{N-1})$. For this case, the lower bound can be obtained numerically which is plotted in Fig. 2 (blue line).

For the next case, consider the Pauli-Y (bit-phase-flip) noise. Calculations in this case show that results are similar to the previous case and therefore both noises have the same effect on the four-qubit W state, namely $\rho_{W_4}^y(t) = \rho_{W_4}^x(t)$.

Now, consider the case that W_4 state is coupled to the Pauli-Z (phase-flip) channel. For this case, the infinitesimal time evolution of the density matrix reads

$$\rho_{W_4}^z(\delta t) = \frac{1}{4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1 & 0 & 0 & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-4\kappa\delta t & 1-4\kappa\delta t & 0 & 1-4\kappa\delta t & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (13)$$

so we take the ansatz

$$\rho_{W_4}^z(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & a & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & b & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

Inserting $\rho_{W_4}^z$ to the Lindblad equation (1) gives rise in

$$\begin{cases} \dot{a}(t) = 0, \\ \dot{b}(t) = -4\kappa b(t), \end{cases} \quad (15)$$

subject to the initial conditions $a(0) = b(0) = 1/4$. Then, we obtain

$$\begin{cases} a(t) = \frac{1}{4}, \\ b(t) = \frac{1}{4}e^{-4\kappa t}. \end{cases} \quad (16)$$

The lower bound in this case is given by

$$\tau_4(\rho_W^z) = e^{-4\kappa t}. \quad (17)$$

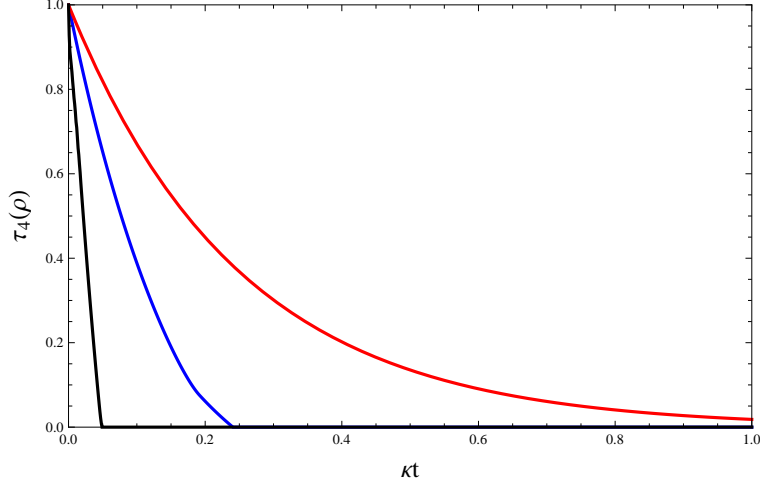


FIG. 2: The lower bound for the four-qubit concurrence for an initial W state transmitted through Pauli-X and Pauli-Y (blue line), Pauli-Z (red line), and isotropic (black line) channels as functions of κt .

This result is depicted in Fig. 2 as a red line.

To this end, we study the interaction of an initial four-qubit W state with the depolarizing channel. Taking into account the effects of all Pauli matrices as noisy operators in the Lindblad equation, the infinitesimal time evolution of the density matrix becomes

$$\rho_{W_4}^d(\delta t) = \frac{1}{4} \begin{pmatrix} 2\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-8\kappa\delta t & 1-12\kappa\delta t & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-12\kappa\delta t & 1-8\kappa\delta t & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\kappa\delta t & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-12\kappa\delta t & 1-12\kappa\delta t & 0 & 1-8\kappa\delta t & 0 & 0 & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\kappa\delta t & 0 & 4\kappa\delta t & 8\kappa\delta t & 0 & 0 & 8\kappa\delta t & 0 & 8\kappa\delta t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8\kappa\delta t & 0 & 8\kappa\delta t & 4\kappa\delta t & 0 & 0 & 0 & 8\kappa\delta t & 0 & 8\kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-12\kappa\delta t & 1-12\kappa\delta t & 0 & 1-12\kappa\delta t & 0 & 0 & 0 & 1-8\kappa\delta t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\kappa\delta t & 0 & 2\kappa\delta t & 0 & 0 & 0 & 4\kappa\delta t & 2\kappa\delta t & 0 & 2\kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 4\kappa\delta t & 0 & 2\kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 0 & 2\kappa\delta t & 2\kappa\delta t & 0 & 4\kappa\delta t & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (18)$$

Therefore, we take the following ansatz:

$$\rho_{W_4}^d = \begin{pmatrix} g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & a & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & e & 0 & c & c & 0 & c & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & a & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & e & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & c & e & 0 & 0 & 0 & c & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & h & 0 & 0 & 0 & d & 0 & d & d & 0 \\ 0 & b & b & 0 & b & 0 & 0 & 0 & a & 0 & 0 & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & c & 0 & 0 & 0 & e & c & 0 & c & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & c & 0 & 0 & c & e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & h & 0 & d & d & 0 \\ 0 & 0 & 0 & 0 & 0 & c & c & 0 & 0 & c & c & 0 & e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & d & 0 & h & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & d & 0 & d & h & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \end{pmatrix}. \quad (19)$$

Inserting this density matrix in Eq. (1), we find

$$\begin{cases} \dot{a}(t) = 2k(g(t) + 3e(t) - 4a(t)), \\ \dot{b}(t) = 4k(c(t) - 3b(t)), \\ \dot{c}(t) = 2k(b(t) + d(t) - 6c(t)), \\ \dot{d}(t) = 4k(c(t) - 3d(t)), \\ \dot{e}(t) = 4k(a(t) + h(t) - 2e(t)), \\ \dot{f}(t) = 8k(h(t) - f(t)), \\ \dot{g}(t) = 8k(a(t) - g(t)), \\ \dot{h}(t) = 2k(3e(t) + f(t) - 4h(t)), \end{cases} \quad (20)$$

in which the initial conditions are $a(0) = b(0) = 1/4$ and $c(0) = d(0) = e(0) = f(0) = g(0) = h(0) = 0$. The solutions are found as

$$\begin{cases} a(t) = \frac{1}{16}(1 + e^{-4\kappa t} + e^{-12\kappa t} + e^{-16\kappa t}), \\ b(t) = \frac{1}{16}e^{-16\kappa t}(1 + e^{4\kappa t})^2, \\ c(t) = \frac{1}{16}e^{-8\kappa t}(1 - e^{-8\kappa t}), \\ d(t) = \frac{1}{16}e^{-16\kappa t}(-1 + e^{4\kappa t})^2, \\ e(t) = \frac{1}{16}(1 - e^{-16\kappa t}), \\ f(t) = \frac{1}{16}(1 - 2e^{-4\kappa t} + 2e^{-12\kappa t} - e^{-16\kappa t}), \\ g(t) = \frac{1}{16}(1 + 2e^{-4\kappa t} - 2e^{-12\kappa t} - e^{-16\kappa t}), \\ h(t) = \frac{1}{16}(1 - 2e^{-4\kappa t} - 2e^{-12\kappa t} + e^{-16\kappa t}). \end{cases} \quad (21)$$

The corresponding lower bound for this state is depicted in Fig. 2 as a black line.

III. EVOLUTION OF ENTANGLEMENT OF MAXIMAL ENTANGLED FOUR-QUBIT STATES

In this section, we compare the time evolution of entanglement of W_4 state with those of maximally entangled four-qubit states, namely Eq. (3). For this purpose, we solve the Lindblad equation (1) for these states and then compute the lower bound to the concurrence for each case.

First, let us consider the state $|\phi_1\rangle$ which is indeed the four-qubit GHZ state. For this case, the Lindblad equation is solved for the Pauli channels σ_x , σ_y , σ_z as well as the isotropic

channel and the lower bound (12) is computed for each case in Ref. [32]. Here, we only present the results as follows:

$$\tau_x(\phi_1) = \tau_y(\phi_1) = \sqrt{2} \max \left\{ 0, \frac{1}{4} (e^{-8\kappa t} + 6e^{-4\kappa t} - 3) \right\}, \quad (22)$$

$$\tau_z(\phi_1) = \sqrt{2} e^{-8\kappa t}, \quad (23)$$

$$\tau_d(\phi_1) = \sqrt{2} \max \left\{ 0, \frac{1}{8} (9e^{-16\kappa t} + 6e^{-8\kappa t} - 7) \right\}, \quad (24)$$

Now, consider the state $|\phi_2\rangle$ which is transmitted through the Pauli-X channel. Its time evolution as a solution of the Lindblad equation (1) and after following the steps similar to the previous section is obtained as

$$\rho_{\phi_2}^x = \begin{pmatrix} a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 \\ a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & c & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b \\ a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 \\ a & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & a & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b \end{pmatrix}, \quad (25)$$

where

$$\begin{cases} a = \frac{1}{16} (1 + e^{-4\kappa t} - 2e^{-6\kappa t}), \\ b = \frac{1}{16} (1 + e^{-4\kappa t} + 2e^{-6\kappa t}), \\ c = \frac{1}{16} (1 - e^{-4\kappa t}). \end{cases} \quad (26)$$

For this case, the lower bound (12) becomes

$$\tau_x(\phi_2) = \sqrt{2} \max \left\{ 0, \frac{1}{2} (2e^{-6\kappa t} + e^{-4\kappa t} - 1) \right\}. \quad (27)$$

For the next case, the state $|\phi_2\rangle$ is exposed to the Pauli-Y channel. The solution reads

$$\rho_{\phi_2}^y = \begin{pmatrix} a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & n & 0 \\ 0 & b & g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & m \\ 0 & g & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & m \\ e & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & n & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & f & 0 & d & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & f & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & c & 0 & d & 0 & 0 & d & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & c & 0 & d & d & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & d & 0 & c & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & c & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & d & 0 & 0 & d & 0 & f & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & d & 0 & f & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & m & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & g \\ n & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & e & 0 \\ n & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & a & 0 \\ 0 & m & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & g & 0 & 0 & b \end{pmatrix}, \quad (28)$$

so that

$$\left\{ \begin{array}{l} a = \frac{1}{16} (1 + e^{-4\kappa t} - 2e^{-6\kappa t}), \\ b = \frac{1}{16} (1 + e^{-4\kappa t} + 2e^{-6\kappa t}), \\ c = \frac{1}{16} (1 - e^{-4\kappa t}), \\ d = \frac{1}{16} (e^{-6\kappa t} - e^{-2\kappa t}), \\ e = -\frac{1}{16} (2e^{-2\kappa t} - e^{-4\kappa t} - e^{-8\kappa t}), \\ f = \frac{1}{16} (e^{-4\kappa t} - e^{-8\kappa t}), \\ g = \frac{1}{16} (2e^{-2\kappa t} + e^{-4\kappa t} + e^{-8\kappa t}), \\ m = \frac{1}{16} (e^{-2\kappa t} + 2e^{-4\kappa t} + e^{-6\kappa t}), \\ n = \frac{1}{16} (e^{-2\kappa t} - 2e^{-4\kappa t} + e^{-6\kappa t}). \end{array} \right. \quad (29)$$

The lower bound (12) now becomes

$$\tau_y(\phi_2) = \sqrt{2} \max \left\{ 0, \frac{1}{8} (e^{-8\kappa t} + 4e^{-6\kappa t} + 6e^{-4\kappa t} + 4e^{-2\kappa t} - 7) \right\}. \quad (30)$$

For transmission of $|\phi_2\rangle$ through the Pauli-Z channel, the time evolution of the corresponding density matrix is

$$\rho_{\phi_2}^z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b \\ 0 & c & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & b \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & a \end{pmatrix}, \quad (31)$$

where

$$\left\{ \begin{array}{l} a = \frac{1}{4}, \\ b = \frac{1}{4} e^{-6\kappa t}, \\ c = \frac{1}{4} e^{-4\kappa t}. \end{array} \right. \quad (32)$$

The lower bound to the concurrence for this case is similar to the lower bound obtained for Pauli-X channel, namely $\tau_z(\phi_2) = \tau_x(\phi_2)$.

Finally, if $|\phi_2\rangle$ is exposed to the depolarizing channel, its density matrix takes the follow-

ing form

$$\rho_{\phi_2}^d = \begin{pmatrix} a & 0 & 0 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & n & 0 \\ 0 & b & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & m \\ 0 & d & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m & 0 & 0 & m \\ e & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & n & n & 0 \\ 0 & 0 & 0 & 0 & c & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c & f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & f & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & c & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & m & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & d \\ n & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & e & 0 \\ n & 0 & 0 & n & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e & a & 0 \\ 0 & m & m & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & b \end{pmatrix}, \quad (33)$$

where

$$\begin{cases} a = \frac{1}{16} (1 + e^{-8\kappa t} - 2e^{-12\kappa t}), \\ b = \frac{1}{16} (1 + e^{-8\kappa t} + 2e^{-12\kappa t}), \\ c = \frac{1}{16} (1 - e^{-8\kappa t}), \\ d = \frac{1}{16} (e^{-8\kappa t} + 2e^{-12\kappa t} + e^{-16\kappa t}), \\ e = -\frac{1}{16} (e^{-8\kappa t} - 2e^{-12\kappa t} + e^{-16\kappa t}), \\ f = \frac{1}{16} (e^{-8\kappa t} - e^{-16\kappa t}), \\ m = \frac{1}{8} (e^{-12\kappa t} + e^{-16\kappa t}), \\ n = \frac{1}{8} (e^{-12\kappa t} - e^{-16\kappa t}). \end{cases} \quad (34)$$

In this case, Eq. (12) gives

$$\tau_d(\phi_2) = \sqrt{2} \max \left\{ 0, \frac{1}{8} (7e^{-16\kappa t} + 8e^{-12\kappa t} - 7) \right\}. \quad (35)$$

Now consider the state $|\phi_3\rangle$. If this state is transmitted through Pauli-X channel, the time evolution of its density matrix is obtained as

$$\rho_{\phi_3}^x = \begin{pmatrix} a & 0 & 0 & d & 0 & d & d & q & 0 & d & d & q & d & q & q & 0 \\ 0 & g & b & 0 & b & 0 & n & m & b & 0 & n & m & n & m & 0 & c \\ 0 & b & g & 0 & b & n & 0 & m & b & n & 0 & m & n & 0 & m & c \\ d & 0 & 0 & p & n & f & f & 0 & n & f & f & 0 & 0 & e & e & u \\ 0 & b & b & n & g & 0 & 0 & m & b & n & n & 0 & 0 & m & m & c \\ d & 0 & n & f & 0 & p & f & 0 & n & f & 0 & e & f & 0 & e & u \\ d & n & 0 & f & 0 & f & p & 0 & n & 0 & f & e & f & e & 0 & u \\ q & m & m & 0 & m & 0 & 0 & l & 0 & e & e & r & e & r & r & 0 \\ 0 & b & b & n & b & n & n & 0 & g & 0 & 0 & m & 0 & m & m & c \\ d & 0 & n & f & n & f & 0 & e & 0 & p & f & 0 & f & 0 & e & u \\ d & n & 0 & f & n & 0 & f & e & 0 & f & p & 0 & f & e & 0 & u \\ q & m & m & 0 & 0 & e & e & r & m & 0 & 0 & l & e & r & r & 0 \\ d & n & n & 0 & 0 & f & f & e & 0 & f & f & e & p & 0 & 0 & u \\ q & m & 0 & e & m & 0 & e & r & m & 0 & e & r & 0 & l & r & 0 \\ q & 0 & m & e & m & e & 0 & r & m & e & 0 & r & 0 & r & l & 0 \\ 0 & c & c & u & c & u & u & 0 & c & u & u & 0 & u & 0 & 0 & h \end{pmatrix}, \quad (36)$$

so that

$$\left\{ \begin{array}{l} a = \frac{1}{48} (3 + 6e^{-4\kappa t} - 8e^{-6\kappa t} - e^{-8\kappa t}), \\ b = \frac{1}{48} (1 + 2e^{-2\kappa t} + 2e^{-4\kappa t} + 2e^{-6\kappa t} + e^{-8\kappa t}), \\ d = \frac{1}{48} (1 + 2e^{-2\kappa t} - 2e^{-6\kappa t} - e^{-8\kappa t}), \\ f = \frac{1}{48} (1 - e^{-8\kappa t}), \\ g = \frac{1}{48} (3 + 4e^{-6\kappa t} + e^{-8\kappa t}), \\ h = \frac{1}{48} (3 + 6e^{-4\kappa t} + 8e^{-6\kappa t} - e^{-8\kappa t}), \\ l = \frac{1}{48} (3 - 4e^{-6\kappa t} + e^{-8\kappa t}), \\ m = \frac{1}{48} (1 - 2e^{-4\kappa t} + e^{-8\kappa t}), \\ n = -\frac{1}{24\sqrt{2}} (1 - e^{-2\kappa t} - e^{-4\kappa t} + e^{-6\kappa t}), \\ c = \frac{1}{24\sqrt{2}} (1 + e^{-2\kappa t} + 3e^{-4\kappa t} + 3e^{-6\kappa t}), \\ e = -\frac{1}{24\sqrt{2}} (1 + e^{-2\kappa t} - e^{-4\kappa t} - e^{-6\kappa t}), \\ q = \frac{1}{24\sqrt{2}} (1 - e^{-2\kappa t} + 3e^{-4\kappa t} - 3e^{-6\kappa t}). \end{array} \right. \quad (37)$$

The lower bound for this density matrix is depicted in Fig. 3(a).

The solution of the Lindblad equation when $|\phi_3\rangle$ is exposed Pauli-Y noise is given by

$$\rho_{\phi_3}^y = \begin{pmatrix} a & 0 & 0 & -d & 0 & -d & -d & q' & 0 & -d & -d & q' & -d & q' & q' & 0 \\ 0 & g & b & 0 & b & 0 & n & -m & b & 0 & n & -m & n & -m & 0 & c' \\ 0 & b & g & 0 & b & n & 0 & -m & b & n & 0 & -m & n & 0 & -m & c' \\ -d & 0 & 0 & p & n & f & f & 0 & n & f & f & 0 & 0 & e' & e' & u \\ 0 & b & b & n & g & 0 & 0 & -m & b & n & n & 0 & 0 & -m & -m & c' \\ -d & 0 & n & f & 0 & p & f & 0 & n & f & 0 & e' & f & 0 & e' & -u \\ -d & n & 0 & f & 0 & f & p & 0 & n & 0 & f & e' & f & e' & 0 & -u \\ q' & -m & -m & 0 & -m & 0 & 0 & l & 0 & e' & e' & r & e' & r & r & 0 \\ 0 & b & b & n & b & n & n & 0 & g & 0 & 0 & -m & 0 & -m & -m & c' \\ -d & 0 & n & f & n & f & 0 & e' & 0 & p & f & 0 & f & 0 & e' & -u \\ -d & n & 0 & f & n & 0 & f & e' & 0 & f & p & 0 & f & e' & 0 & -u \\ q' & -m & -m & 0 & 0 & e' & e' & r & -m & 0 & 0 & l & e' & r & r & 0 \\ -d & n & n & 0 & 0 & f & f & e' & 0 & f & f & e' & p & 0 & 0 & -u \\ q' & -m & 0 & e' & -m & 0 & e' & r & -m & 0 & e' & r & 0 & l & r & 0 \\ q' & 0 & -m & e' & -m & e' & 0 & r & -m & e' & 0 & r & 0 & r & l & 0 \\ 0 & c' & c' & u & c' & -u & -u & 0 & c' & -u & -u & 0 & -u & 0 & 0 & h \end{pmatrix}, \quad (38)$$

where $a, b, d, f, g, h, l, m, n, p, r$ and u are given in Eq. (37) and c', e' and q' are found as

$$\left\{ \begin{array}{l} c' = \frac{1}{24\sqrt{2}} (3e^{-2\kappa t} + 3e^{-4\kappa t} + e^{-6\kappa t} + e^{-8\kappa t}), \\ e' = -\frac{1}{24\sqrt{2}} (e^{-2\kappa t} + e^{-4\kappa t} - e^{-6\kappa t} - e^{-8\kappa t}), \\ q' = \frac{1}{24\sqrt{2}} (3e^{-2\kappa t} - 3e^{-4\kappa t} + e^{-6\kappa t} - e^{-8\kappa t}). \end{array} \right. \quad (39)$$

The lower bound for this density matrix is shown in Fig. 3(b).

Now, let us consider the state $|\phi_3\rangle$ which is transmitted through Pauli-Z channel. After

solving the Lindblad equation, the time evolution of the density matrix is obtained as

$$\rho_{\phi_3}^z = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & b & a & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & a & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & b & 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & c & c & 0 & c & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & d \end{pmatrix}, \quad (40)$$

where

$$\begin{cases} a = \frac{1}{6}, \\ b = \frac{1}{6}e^{-4\kappa t}, \\ c = \frac{1}{3\sqrt{2}}e^{-6\kappa t}, \\ d = \frac{1}{3}. \end{cases} \quad (41)$$

Computation of the lower bound gives rise to

$$\tau_z(\phi_3) = \frac{2}{3}\sqrt{(3e^{-12\kappa t} + e^{-8\kappa t})}. \quad (42)$$

To this end, consider the transmission of $|\phi_3\rangle$ through the depolarizing channel. In this case, the time evolution of the state is described by

$$\rho_{\phi_3}^d = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & e & 0 & 0 & 0 & e & 0 & e & e & 0 \\ 0 & d & b & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & b & d & 0 & b & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & g & 0 & m & m & 0 & 0 & m & m & 0 & 0 & 0 & 0 \\ 0 & b & b & 0 & d & 0 & 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & m & 0 & g & m & 0 & 0 & m & 0 & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 & m & g & 0 & 0 & 0 & m & 0 & m & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & n & 0 & n & 0 \\ 0 & b & b & 0 & b & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & m & 0 & m & 0 & 0 & 0 & g & m & 0 & m & 0 & 0 \\ 0 & 0 & 0 & m & 0 & 0 & m & 0 & 0 & m & g & 0 & m & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & 0 & f & 0 & n & 0 \\ 0 & 0 & 0 & 0 & 0 & m & m & 0 & 0 & m & m & 0 & g & 0 & 0 \\ e & 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & 0 & n & 0 & f & n \\ e & 0 & 0 & 0 & 0 & 0 & 0 & n & 0 & 0 & 0 & n & 0 & f & 0 \\ 0 & c & c & 0 & c & 0 & 0 & 0 & c & 0 & 0 & 0 & 0 & 0 & h \end{pmatrix}, \quad (43)$$

so that

$$\begin{cases} a = \frac{1}{48}(3 + 6e^{-8\kappa t} - 8e^{-12\kappa t} - e^{-16\kappa t}), \\ b = \frac{1}{24}(e^{-8\kappa t} + 2e^{-12\kappa t} + e^{-16\kappa t}), \\ c = \frac{1}{6\sqrt{2}}(e^{-12\kappa t} + e^{-16\kappa t}), \\ d = \frac{1}{48}(3 + 4e^{-12\kappa t} + e^{-16\kappa t}), \\ e = \frac{1}{6\sqrt{2}}(e^{-12\kappa t} - e^{-16\kappa t}), \\ f = \frac{1}{48}(3 - 4e^{-12\kappa t} + e^{-16\kappa t}), \\ h = \frac{1}{48}(3 + 6e^{-8\kappa t} + 8e^{-12\kappa t} - e^{-16\kappa t}), \\ m = \frac{1}{24}(e^{-8\kappa t} - e^{-16\kappa t}), \\ n = \frac{1}{24}(e^{-8\kappa t} - 2e^{-12\kappa t} + e^{-16\kappa t}). \end{cases} \quad (44)$$

The lower bound to the concurrence is plotted in Fig. 3(d).

In Fig. (3), we plotted the results for the four investigated states. It should be noted that although the states introduced in Eq. (3) are maximally entangled, the initial lower bound to the concurrence for these states gives rise to different values, i.e., $\tau(\phi_1) = \tau(\phi_2) = \sqrt{2}$ and $\tau(\phi_3) = 4/3$. So, we normalized them so that $\tau(\phi_1) = \tau(\phi_2) = \tau(\phi_3) = 1$ at $\kappa t = 0$. As it is shown in Fig. (3), the entanglement present in W_4 state is more robust than the other investigated states against decoherence under bit-flip, bit-phase-flip, and phase-flip noises. However, the four-qubit GHZ state is more robust in the presence of the depolarizing noise.

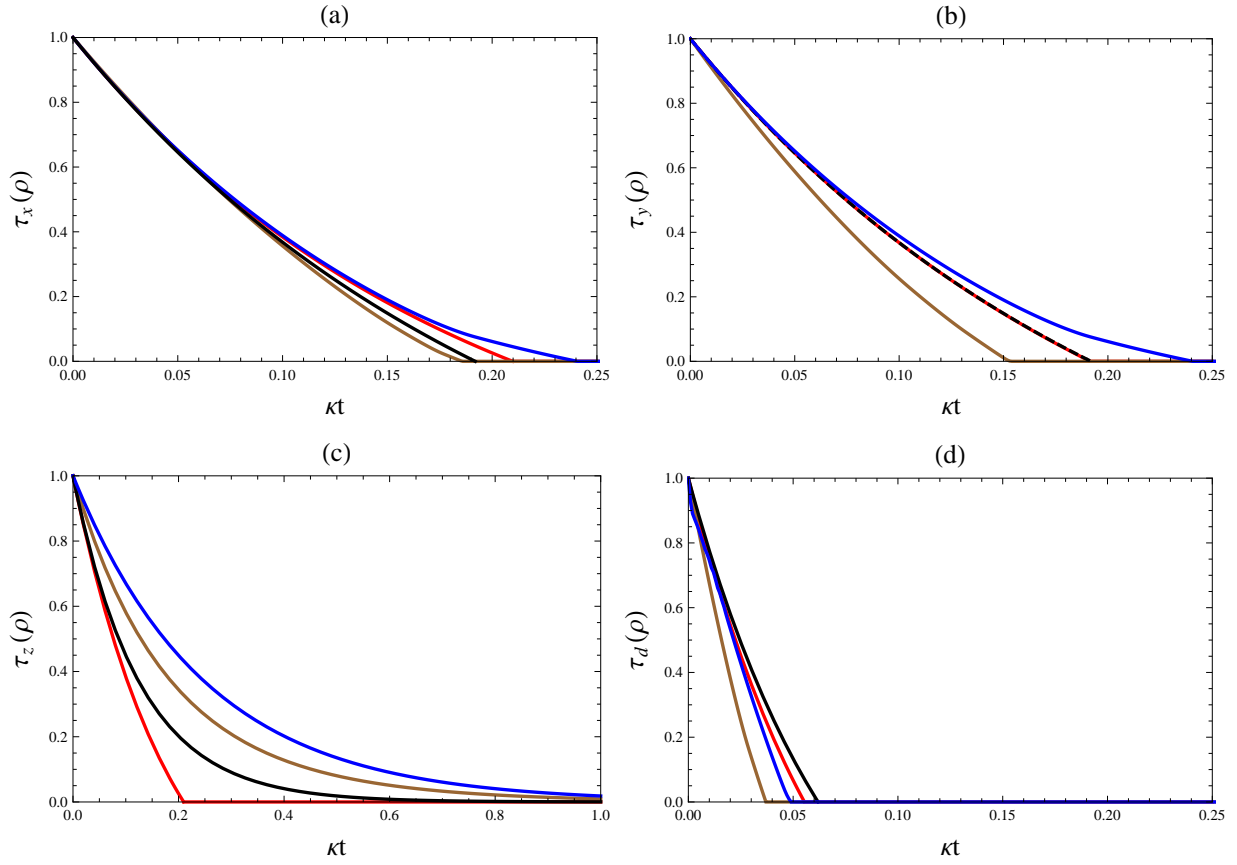


FIG. 3: The lower bound to the concurrence for W_4 (blue line), four-qubit GHZ (black line), ϕ_2 (red line) and ϕ_3 (brown line) under: (a) Pauli-X (b) Pauli-Y (c) Pauli-Z and (d) isotropic noisy channels versus κt .

IV. CONCLUSIONS

In this paper, we have investigated the dynamics of an initial four-qubit W state in interaction with its surrounding environment. We exactly solved the Lindblad equation in which the Lindblad operators are proportional to the Pauli matrices and obtained the density matrices corresponding to each noisy channel. We also examined the time evolution of entanglement using the lower bound to the concurrence for four-qubit W state. It is found that the entanglement of states vanishes after some finite time for the Pauli channels σ_x and σ_y as well as the isotropic channel. However, for the Pauli-Z noise, the entanglement exponentially decreases and vanishes asymptotically. Moreover, by exactly solving the Lindblad equation, we studied the effects of various noises on three four-qubit maximally entangled states and obtained the time evolution of the lower bound. We found that except the depolarizing channel, the W_4 state is more robust against the decoherence with respect to ϕ_1 , ϕ_2 , and ϕ_3 . Also, except Pauli-Z channel, W_3 is more robust than W_4 under the noises [25].

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